Technical Note

Golden Equations for Dynamic Characteristics of Beam-Spring Systems

Abdolrasoul Ranjbaran[1*] and Mohammad Ranjbaran[2]

1. Associate Prof., Department of Civil Engineering, Shiraz University, Shiraz, Iran,
* Corresponding Author; e-mail: ranjbarn@shirazu.ac.ir
2. MSc Candidate, Department of Chemical Engineering, Shiraz University, Shiraz, Iran

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ABSTRACT

In damage detection of civil, mechanical, aerospace, nuclear, bio-mechanic, and offshore engineering, dynamic characteristics of beam-spring structures (BSS) play an important role. Recently, a new and innovative method for the free vibration of cracked bars is proposed. The method is extended for the free vibration of BSS. By introducing a new and innovative conjugate beam through defining a new variable, a single ordinary differential equation, golden equation, is obtained. The solution for the golden governing equation (GGE) is the same as that for an intact beam and so great simplicity and generality is obtained. Using the GGE, both closed form and numerical solutions are obtained. Through applying the work to specific examples and comparison of the results with the others, the accuracy, efficiency and robustness of the work is verified.

Keywords:
Beam-spring system; Dynamic characteristics; Crack; Ranjbaran transformation; Stiffness Reduction method; Golden equation; Golden derivative

1. Introduction

In order to assess the integrity and serviceability of existing structures, the need for development of efficient procedures to be used in non-destructive structural damage detection is increasing. In lieu of this, the research for development of methods that could identify changes in dynamic characteristics of a structural system is continued. These methods are based on the fact that modal parameters (notably frequencies and mode shapes, and modal damping) are functions of the physical properties of the system (mass, damping and stiffness). Any change in the physical properties, such as reduction in stiffness resulting from cracking or loosening of a connection, will cause detectable changes in the modal properties. Obtaining the explicit solution for free vibration analysis of the beam spring systems (BSS), Figure (1), is difficult. The problem of detecting, locating, and quantifying the extent of damage was under study for several decades [1-2]. Patil and Maiti [3] developed a method for detection of position and extent of cracks in a beam from measured dynamic characteristics. They verified their work by experimental results in [4].

Shifrin and Ruotolo [5] proposed a technique for calculating natural frequencies of a vibrating beam with an arbitrary finite number of transverse open cracks. They used rotational mass-less spring for crack model. Their paper has a good mathematical foundation. Behzad et al [6], based on Hamilton principle, developed the equation of motion and corresponding boundary conditions for bending vibration of a beam with an open edge crack. The natural frequencies of a uniform Euler-Bernoulli beam have been calculated using the new developed model in conjunction with the Galerkin projection method. Binici [7] studied vibration of beams with multiple open cracks subjected to axial force. He proposed a method to obtain the eigen-frequencies and eigen-modes of beams containing multiple cracks and subjected to axial force. Within the framework of the distribution theory, a model for treatment of vibrating beams in the presence of multiple open

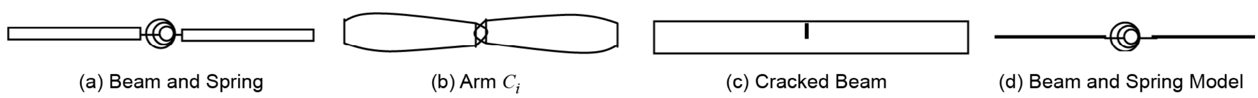


Figure 1. Beam and spring model.

concentrated cracks is studied by Caddemi and Calio [8].

A new and innovative method for determining the dynamic characteristics of cracked bars is presented in Ranjbaran et al [9] and for beams in Ranjbaran [10]. The latter is recently cited in Caddemi and Calio [14]. The method is extended for determining the dynamic characteristics of BSS and in the investigation on the dynamic stability analysis of BSS subjected to axial force in this paper.

A structure is designed to withstand a specific earthquake. The effect of earthquake on the structure is a function of its natural frequencies, frequency content of the soil and the rock under the structure [15]. Cracks and other types of damages change the frequencies. The change in frequencies may change the response of the structure to earthquake. This change should be computed; its effect on the performance of the structure is to be investigated; and the structure is to be retrofitted if necessary. This shows a strong relation between this study and the earthquake engineering. Moreover, the earthquakes introduce dynamic instability into the structures. The present work may be used as a base for the investigation of the dynamic stability of imperfect structures.

2. Free Vibration by Equivalent Mass Method (EMM)

2.1. The Governing Equations

Based on the principles of fracture mechanics, a crack introduces a jump in the tangent to the lateral displacement [11]. The governing equation (GE) for free vibration and the compatibility conditions (C.C.) at a cracked point are defined as:

$$GE: EIy^{IV} - m\omega^2 y = 0 \tag{1}$$

and

$$C.C.: \{ y_i^+ = y_i^-, \; y_i'^+ - y_i'^- = C_{bi} y_i'', \; y_i''^+ = y_i''^-, \; y_i'''^+ = y_i'''^- \} \tag{2}$$

respectively. In which E, I, y, m, ω, and C_{bi} are elastic modulus, 2nd moment of area, lateral displacement, mass per unit length, natural frequency and crack compliance, respectively. The (') in the superscript denotes differentiation with respect to member axis, (+) and (−) denote the right and left point of the cracked point, respectively. Conventionally, the closed form solution (CFS) is obtained in the following steps. The member is divided into segments connected at the nodes. The crack is positioned on a node. The solution for the GE is written for each segment in terms of four unknown coefficients. Applying the (C.C.) at the nodes and the boundary conditions (B.C.) at the end points leads to an eigen system of equations. The size of the system increases with the number of cracks. The problem soon becomes cumbersome. This is the reason for solution of beams with few cracks to date. The authors of present paper tried to combine the GE and the continuity conditions at the cracked point and introduce a new GE for the cracked member, which needs only the (B.C.) for its solution. Toward reaching the goal, the jump is defined as a new variable, y'_{crack} as follows [10]:

$$y'_{crack} = \sum_{i=1}^{n_c} C_{bi} y_i'' H_i, \; H_i = H(x - x_i) \tag{3}$$

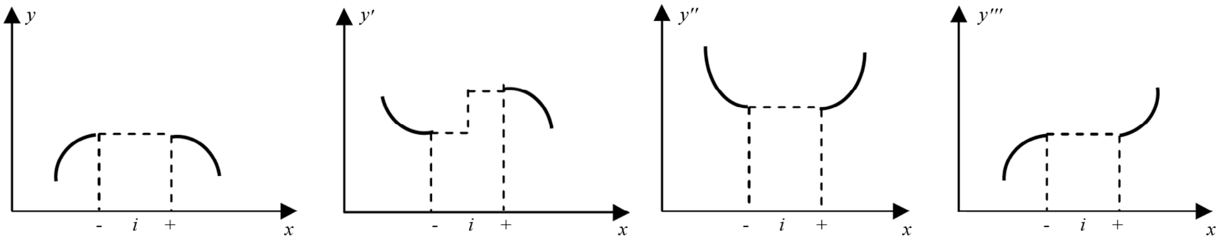
where n_c, y_i'' and H_i are the number of cracks, displacement curvature and the Heaviside unit step function, respectively. The curvatures on both sides of a crack are equal, Figure (2). As a result, the derivative of curvature at this point is zero. In lieu of that, the derivative of Eq. (3) is written as follows:

$$y''_{crack} = \sum_{i=1}^{n_c} C_{bi} y_i'' \delta_i, \; \delta_i = \delta(x - x_i) \tag{4}$$

in which δ is the Dirac delta distribution. This is called the golden derivative. It paved the way for a novel and innovative formulation. The GE for free vibration analysis of intact beam is rewritten as:

$$y^{IV} - \lambda^4 y = 0, \; EI\lambda^4 = m\omega^2 \tag{5}$$

where λ is a working parameter. Combining the GE (5) and the (C.C.) (4) is not possible, because their differentiation orders are different. The problem is solved as follows. Twice integration of Eq. (5) leads to the following equation:



(a) $y_{-i} = y_{+i}$ (b) $y'_{+i} - y'_{-i} = C_i y''_i$ (c) $y''_{-i} = y''_{+i}$ (d) $y'''_{-i} = y'''_{+i}$

Figure 2. Continuity condition at a cracked point for flexure y'.

$$y'' - \lambda^4\left(\iint y dx dx + b_1 x + b_2\right) = 0 \tag{6}$$

The terms in braces is defined as new variable w. In terms of the new variables, the GE (6) is written as follows:

$$w^{IV} - \lambda^4 w = 0,\ w = \iint y dx dx + b_1 x + b_2, \\ y = w'',\ w^{IV} = \lambda^4 w \tag{7}$$

and the golden derivative, Eq. (4), as:

$$w^{IV}_{crack} = \sum_{i=1}^{n_c} C_{bi} w_i^{IV} \delta_i = \lambda^4 \sum_{i=1}^{n_c} C_{bi} w_i \delta_i \tag{8}$$

Assume that w^{IV} is the sum of the intact w^{IV}_{intact} and crack w^{IV}_{crack}. Substitute the intact term, as the difference between the total and the cracked term, into Eq. (7) to obtain the golden governing equation (GGE) for a BSS as follows:

$$w^{IV} - \lambda^4\left(1 + \sum_{i=1}^{n_c} C_{bi} w_i \delta_i\right) w = 0 \tag{9}$$

The zero boundary conditions corresponding to w are obtained in terms of y, Table (1), as follows:

$$w_e = y''_e,\ w'_e = y'''_e,\ w''_e = y_e,\ w'''_e = y'_e \tag{10}$$

The GE for the free vibration of a beam with a point mass, M_i, is:

$$z^{IV} - \lambda^4(1 + M_i \delta_i / m) z = 0 \tag{11}$$

where z is the lateral displacement. Eqs. (9) and (11) are equal except in the name of parameters. The similarity shows that the Eq. (9) is the GE for the free vibration of beams with a concentrated mass, $M_i = mC_{bi}$, at the crack position. Based on this observation, the method is called the equivalent mass method (EMM). The EMM is a special transformation. It is called Ranjbaran transformation. In this transformation, the crack beam is replaced by an intact one and the crack is replaced by an equivalent point mass, and the end conditions are modified as in Table (1).

2.2. The Closed Form Solution

The Laplace Transform (LT) [12] is considered as the best method for the closed form solution (CFS) of the GGE (9). The LT of Eq. (9) is defined as follows:

$$s^4 L_w - s^3 w_0 - s^2 w'_0 - s w''_0 - \\ w'''_0 - \lambda^4\left(L_w + \sum_{i=1}^{n_c} C_{bi} e^{-sx_i} w_i\right) = 0 \tag{12}$$

Table 1. End conditions in two coordinates.

End Conditions					
Y-Coordinate			W-Coordinate		
Name	Equations	Shape	Shape	Equations	Name
Simple	$y = 0$, $y'' = 0$			$w'' = 0$, $w = 0$	Simple
Fixed	$y = 0$, $y' = 0$			$w' = 0$, $w''' = 0$	Free
Free	$y'' = 0$, $y''' = 0$			$w = 0$, $w' = 0$	Fixed

or

$$L_w = \frac{s^3 w_0}{s^4 - \lambda^4} + \frac{s^2 w_0'}{s^4 - \lambda^4} + \frac{s w_0''}{s^4 - \lambda^4} + \frac{w_0'''}{s^4 - \lambda^4} + \lambda^4 \sum_{i=1}^{n_c} C_{bi} \frac{e^{-sx_i} w_i}{s^4 - \lambda^4} \quad (13)$$

By taking the inverse LT of Eq. (13), the CFS is obtained as follows:

$$w = w^*(x) + \lambda \textstyle\sum_{i=1}^{n_c} C_{bi} w(x_i)\, f_4\, H_i$$
$$w^*(x) = w_0 f_0(x) + w_0' f_1(x) + w_0'' f_2(x) + w_0''' f_3(x) \quad (14)$$

where

$$2 f_0(x) = (\cos\lambda x + \cosh\lambda x)$$
$$2 f_1(x) = (\sin\lambda x + \sinh\lambda x)\lambda^{-1} \quad (15)$$
$$2 f_2(x) = (-\cos\lambda x + \cosh\lambda x)\lambda^{-2}$$

$$2 f_3(x) = (-\sin\lambda x + \sin h\lambda x)\lambda^{-3}$$
$$2 f_4(x) = (-\sin\lambda(x - x_i) + \sin h\lambda(x - x_i)) \quad (16)$$

The three out of four unknown coefficients $(w_0, w_0', w_0'', w_0''')$ of Eqs. (14) and λ are to be determined by inserting the four boundary conditions. The exact solution (mode shape) is then determined in terms of one coefficient. The actual mode shape (i.e. y) is determined from that for w by differentiation.

2.3. The Finite Element Solution

To determine the finite element equation corresponding to the GGE (9) its weighted residual is set equal to zero, then the chain rule is used to change the error equation into the weak form (WF) as follows [13]:

$$\int_0^L \psi \left[w^{IV} - \lambda^4 \left(1 + \sum_{i=1}^{n_c} C_{bi} w_i \delta_i \right) w \right] dx = 0$$
$$\rightarrow \int_0^L \psi'' w'' dx - \lambda^4 \left(\int_0^L \psi w dx + \sum_{i=1}^{n_c} C_{bi} \psi_i w_i \right) = 0 \quad (17)$$

Define the weight function and the main parameter in terms of the nodal values as follows:

$$\psi = \psi^\alpha N_\alpha,\ w = w^\beta N_\beta,\ N''_{\alpha\beta} = N''_\alpha N''_\beta,$$
$$N_{\alpha\beta} = N_\alpha N_\beta,\ \alpha,\beta = 1, n_d \quad (18)$$

In this equation, the Einstein's summation convention is used on the number of element degrees of freedom number n_d. Substitute Eq. (18) into the WF to obtain the finite element equation as follows:

$$[k^o_{\alpha\beta} - \lambda^4 (m_{\alpha\beta} + m^{eq}_{\alpha\beta})]\, w^\beta = 0,\ k^o_{\alpha\beta} =$$
$$\int_0^L N''_{\alpha\beta}\, dx,\ m_{\alpha\beta} = \int_0^L N_{\alpha\beta}\, dx,\ m^{eq}_{\alpha\beta} = \sum_{i=1}^{n_c} C_{bi} N_{\alpha\beta}(x_i) \quad (19)$$

where $k^o_{\alpha\beta}, m_{\alpha\beta}$ and $m^{eq}_{\alpha\beta}$ are the stiffness, the mass and the equivalent mass matrices, respectively. Note that Eq. (19) is valid for both a member and an element.

3. Free Vibration By Stiffness Reduction Method (SRM)

3.1. The Governing Equation

The method presented in the previous section is accurate and simple both in derivation and in implementation. The EMM is not suitable for the general dynamic and static analysis of the cracked beam and buckling analysis of the cracked columns. Toward the aim, the GGE is obtained by inserting Eq. (4) into Eq. (5) as follows:

$$\left(y'' - \sum_{i=1}^{n_c} C_{bi} y_i'' \delta_i \right)'' - \lambda^4 y =$$
$$y^{IV} - \sum_{i=1}^{n_c} C_{bi} (y_i'' \delta_i)'' - \lambda^4 y = 0 \quad (20)$$

3.2. The Closed form Solution

The LT of Eq. (20) is written as follows:

$$s^4 L_y - s^3 y_0 - s^2 y_0' - s y_0'' - y_0''' - \lambda^4 L_y -$$
$$\sum_{i=1}^{n_c} C_{bi} \left[s^2 L(y_i''\delta_i) - s(y_i''\delta_i)_0 - (y_i''\delta_i)'_0 \right] = 0 \quad (21)$$

The LT of y is obtained as follows:

$$L_y = \frac{s^3 y_0}{s^4 - \lambda^4} + \frac{s^2 y_0'}{s^4 - \lambda^4} + \frac{s y_0''}{s^4 - \lambda^4} + \frac{y_0'''}{s^4 - \lambda^4} + \sum_{i=1}^{n_c} C_{bi} \frac{s^2 e^{-sx_i} y_i''}{s^4 - \lambda^4} \quad (22)$$

Take the inverse LT from Eq. (22) to obtain the closed form solution as follows:

$$y = y^*(x) + \lambda^{-1} \textstyle\sum_{i=1}^{n_c} C_{bi} y''(x_i)\, f_5 H_i,$$
$$2 f_5(x) = (\sin\lambda(x - x_i) + \sinh\lambda(x - x_i)) \quad (23)$$
$$y^*(x) = y_0 f_0(x) + y_0' f_1(x) + y_0'' f_2(x) + y_0''' f_3(x)$$

3.3. The Finite Element Equation

The weighted residual method is applied to the GGE (20) as follows:

$$\int_0^L \psi\left[y^{IV} - \sum_{i=1}^{n_c} C_{bi}(y_i''\delta_i)'' - \lambda^4 y \right] dx = 0$$
$$\rightarrow \int_0^L \psi'' y'' dx - \sum_{i=1}^{n_c} C_{bi}\psi_i'' y_i'' - \lambda^4 \int_0^L \psi\, y\, dx = 0\ (WF) \tag{24}$$

Write the weight and the displacement in terms of the nodal values and insert into the WF to obtain the finite element equation as follows:

$$(k^o_{\alpha\beta} - k^{eq}_{\alpha\beta} - \lambda^4 m_{\alpha\beta})\, y^\beta = 0,\ k^o_{\alpha\beta} = \int_0^{L_e} N''_{\alpha\beta}\, dx,\ k^{eq}_{\alpha\beta} = \sum_{i=1}^{n_c} C_{bi} N''_{\alpha\beta}(x_i),\ m_{\alpha\beta} = \int_0^{L_e} N_{\alpha\beta}\, dx \tag{25}$$

where $k^{eq}_{\alpha\beta}$ is the equivalent stiffness, L_e is the element length and n_e is the number of elements. In deriving the Eq. (25), the conventional procedure as used for deriving the Eq. (19) is applied. The implementation of Eq. (25) and solving typical problems proves it to be erroneous. Based on extensive numerical experiments and curve fitting the correct form for the equivalent stiffness is defined as follows:

$$k^{eq}_{\alpha\beta} = \sum_{i=1}^{n_c} C_{bi} L_e N''_{\alpha\beta}(x_i) / (L_e + 4C_{bi}) \tag{26}$$

Based on the above formula, a finite element program is developed. The program is used for numerical solutions.

The authors derived the equivalent stiffness in Eq. (26). It is proved based on numerical experiments and physical concerns. Deriving the correct form for the finite element Eq. (25) mathematically is under study but is not finished yet.

4. Verification

To verify the formula and its implementation, an example is included.

Example: Free Vibration of a Fixed-Free Multi-Cracked Beam

A fixed-free beam of $L = 0.8$ m and square cross-section, $b = h = 0.02$ m is selected for study. The material properties are assumed as Young's modulus $E = 2.1 \times 10^{11}$ N/m^2, Poisson's ratio $v = 0.35$ and $\rho = 3800$ kg/m^3 mass density. The beam has two cracks at $x_1 = 0.12$ m with $a_1 = 2$ mm and $x_2 = 0.4$ m with $a_2 = 3$ mm, respectively, see Figure (3).

Solution: The beam is modeled by 20 equal length beam element. The model is solved by the EMM, Figure (4) and SRM, Figure (5). This example is solved in Shifrin and Ruotolo [5]. The above solutions along with that of ANSYS are compared in Figure (6) and included in Table (2) for completeness. The results are in an excellent agreement. This example shows that the present work is more accurate and at the same time much easier than the others.

Dynamic stability is an important issue in the earthquake engineering investigations. Recently, the authors presented their work regarding this issue [16-17]. One of the governing equations for the dynamic stability of a column is of the following form:

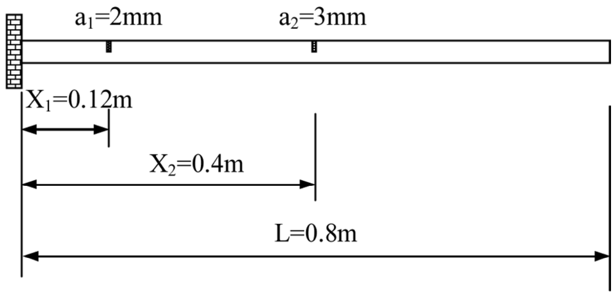


Figure 3. The fixed-free cracked beam.

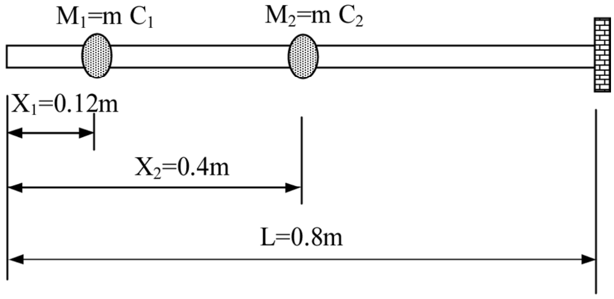


Figure 4. The EMM model.

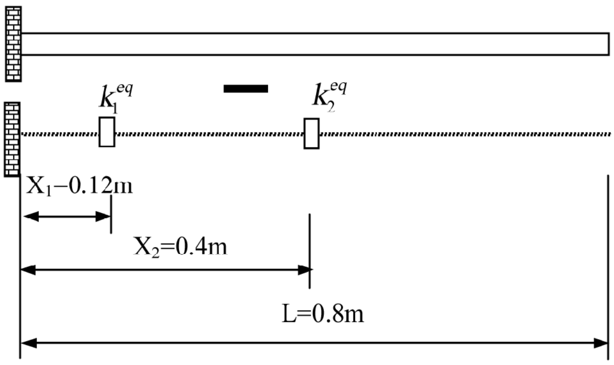


Figure 5. The SRM model.

Table 2. Natural frequencies of a beam with two cracks.

Method	Frequency					
	f_1	f_2	f_3	f_4	f_5	f_6
ANSYS	26.198	163.690	456.15	887.770	1459.900	1622.300
Ref. [5]	26.095	163.322	459.601	-	-	-
CFS	26.095	163.320	459.601	895.132	1486.443	2247.501
EMM	26.095	163.323	459.608	895.900	1486.667	2210.957
SRM	26.088	163.125	459.606	894.599	1486.434	2206.454

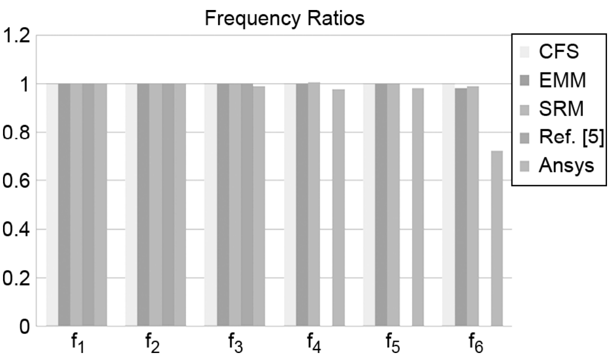


Figure 6. Frequency Ratios for example 1 (L = 0.8m).

$$T_*(R_\omega, R_P) = 2R_\omega^2 + (R_P^2 + 2R_\omega^2)\cos\alpha L \cos h\beta L - R_P R_\omega \sin\alpha L \sin h\beta L = 0 \tag{27}$$

in which the parameters are defined as follows:

$$R_\omega^2 = \frac{mL^4\omega^2}{\pi^4 EI},\ R_P = \frac{PL^2}{\pi^2 EI},$$
$$2\alpha^2 L^2 = \pi^2\left(\sqrt{R_P^2 + 4R_\omega^2} + R_P\right),$$
$$2\beta^2 L^2 = \pi^2\left(\sqrt{R_P^2 + 4R_\omega^2} - R_P\right) \tag{28}$$

Investigation of the dynamic stability needs the solution of the non-linear Eq. (27). Based on the equations presented in this paper, solution of the Eq. (27) for the cases $(b = R_\omega,\ \text{for}\ R_P = 0)$ and $(a = R_P,\ \text{for}\ R_\omega = 0)$ may be obtained. Using the results, the solution for the Eq. (27) is easily obtained from the following equation:

$$\frac{R_P}{a} + \frac{R_\omega^2}{b^2} = 1 \tag{29}$$

Example 2: Dynamic stability of a cantilever column.

A cantilever column of length $L=15$ and $b=h=1$ cross sectional area is considered. The material properties are assumed as Young's modulus $E=2.1 \times 10^{11}$ N/m^2, Poisson's ratio $v=0.35$ and $\rho=7800$kg/m^3 mass density. The column has one crack at $x=0$ with crack depth $a=0.5$, $\xi = a/h$.

Solution: The solution for the intact and cracked column using both Eqs. (27) and (29) is shown in Figure (7). In this figure, ξ_N denotes numerical solution and ξ_A denotes analytical solution. Excellent agreement of the results verified the work.

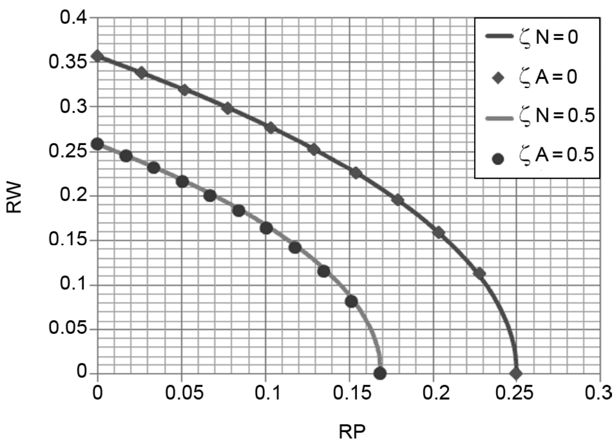


Figure 7. The relation for the column in example 2.

5. Conclusions

The following conclusions are obtained from this study:

- Governing equations for free vibration analysis of cracked BSS are derived.
- The effect of springs is specified by special derivatives called golden derivatives. These derivatives paved the way for present form of formulation.
- Based on the governing equations, finite element equations for analysis of cracked BSS are developed. The elements are in the standard finite element format. The elements are the most simple

and accurate finite elements developed for the problem to date in the literature.

- The global effect of singularity introduced by a crack is changed to an effect in the element level by introducing a multiplier. Based on extensive numerical experimentation, an exact and simple formula for the multiplier is defined. More theoretical derivation is underway.
- The present formulation paved the way for simple and accurate analysis of framed structures containing BSS members.
- For the simple formulation of the dynamic stability of columns, the presented formula can be used.

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